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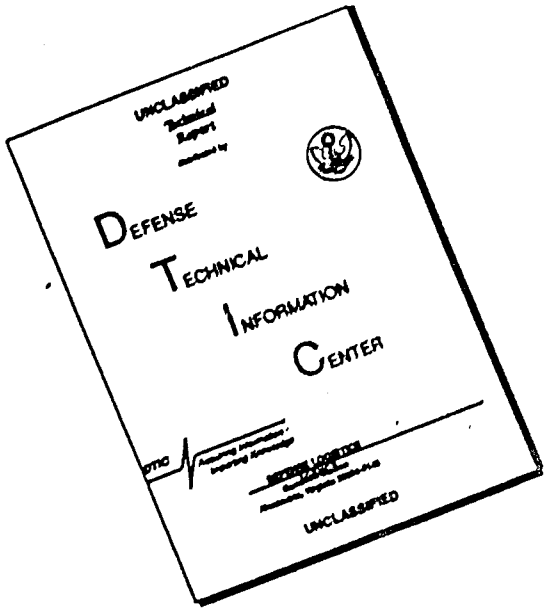

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Technical Report No. 8

THE THICK-WALLED HOLLOW SPHERE OF AN ELASTIC-LOCKING MATERIAL

by

Aris Phillips

and

Asim Yildiz

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DEPARTMENT OF CIVIL ENGINEERING

YALE UNIVERSITY

NEW HAVEN, CONN.

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I INTRODUCTION

The purpose of this study is the exploration of the rheological concept of the locking material. This concept has been introduced by Prager[1]. In the one dimensional case it can be represented by means of the stress-strain diagram in Fig. 1. It is seen that at a given value of the strain, $\varepsilon = \xi_1$, the stress-strain curve becomes a straight line parallel to the σ -axis.

In another paper[2] Prager introduced the concept of the elastic solid of limited compressibility. Such a solid is linearly elastic as long as the mean pressure remains below a certain critical value. As the critical pressure is exceeded, the mean compression remains at the level associated with the critical pressure, while the ratio between corresponding components of the deviations of stress and strain continues to have the elastic value 2G. This solid can be represented by means of the two curves in Figs. 2a and 2b.

By a synthesis of the two concepts introduced by Prager previously, Phillips[3] introduced the concept of the ideal locking material which is represented by means of the two curves in Figs. 3a and 3b. It is seen that such a material may lock either by compression or by shear.

In this study the work of Prager and Phillips will be extended by introducing the elastic-locking material represented by Figs. 4a and 4b. This material is an extension of the ideal locking material. This

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(1) W. Prager: "On Ideal Locking Materials", Trarsactions of the Society of Rheology, vol. I, (1957), pp. 169-175.

(2) W. Prager: "Elastic Solids of Limited Compressibility", Proceedings IX International Congress of Applied Mechanics, vol. 5, (1957), pp. 205-211.

(3) A. Phillips: "A Theory of Ideal Locking Materials", Technical Report No. 4 to Office of Naval Research, Contract Nonr - 60912, February 1958. To appear in vol. III of the Transactions of the Society of Rheology.

extension, however, introduces considerable complications in the solution of problems.

The study of the elastic-locking material will give us information about the correlation between locking in shear and locking in hydrostatic compression, when the respective diagrams posses elastic regions. A procedure will be shown for solving boundary value problems for such a material. This procedure differs in some respects from the procedures used in elasticity or plasticity because of the possibility of locking either in shear or in hydrostatic compression. This procedure will be explained by solving the problem of the thick-walled hollow sphere under internal and external pressures.

In the rest of this paper the following notations will be used:

$$\sigma_m = (\sigma_r + \sigma_\theta + \sigma_\varphi)/3$$

$$\varepsilon_m = (\varepsilon_r + \varepsilon_\theta + \varepsilon_\varphi)/3$$

$$\gamma_0 = \frac{2}{3}\left[(\varepsilon_r - \varepsilon_\theta)^2 + (\varepsilon_\theta - \varepsilon_\varphi)^2 + (\varepsilon_\varphi - \varepsilon_r)\right]^{1/2}$$

II THE THICK-WALLED HOLLOW SPHERE

In this study we shall consider the problem of the thick-walled hollow sphere under internal and external pressures. The material of this sphere will be of the following type. It is assumed that for the change in volume the material follows the law shown in Fig. 5. For the change in shape, it is assumed that there exists a locking condition, $g = \gamma_0 = k = \text{constant}$, so that for values $\gamma_0 < k$ the change in shape follows Hooke's law, whereas for $\gamma_0 = k$ the change in shape follows the stress-strain relations:

$$\sigma_i - \sigma_m = \lambda \frac{\partial \gamma_0}{\partial (\varepsilon_i - \varepsilon_m)}$$

$$\tau_{ij} = \lambda \frac{\partial \gamma_0}{\partial (\gamma_{ij}/2)} \qquad (1)$$

where λ is a proportionality factor. Finally we can never have $\gamma_0 > k$.

According to the terminology introduced in the previous article this is a locking material with limited compressibility. As this material follows Hooke's law when it is not locked, we shall call it an elastic-locking material.

In what follows the hollow sphere will have the internal radius a and the external radius b. The external pressure will be denoted by $-\sigma_{rb}$ and the internal pressure by $-\sigma_{ra}$. The displacements of the particles of the sphere will be radial and those of the internal surface will be denoted by u_a whereas those of the external surface will be denoted by u_b.

We obviously have the following four cases:

1. $\varepsilon_m < k_m$ and $\gamma_0 < k$; this means that the material follows Hooke's law.

2. $\varepsilon_m < k_m$ and $\gamma_0 = k$; this means that the change in volume is elastic while the change in shape occurs under conditions of distortional locking.

3. $\varepsilon_m = k_m$ and $\gamma_0 < k$; that is volumetric locking occurs while the change in shape is elastic.

4. $\varepsilon_m = k_m$ and $\gamma_0 = k$; there is simultaneous volumetric and distortional locking.

It should be remembered that at a given state of loading only one of these four cases can materialize at a given point of the sphere. However, for the same load but at two different points two different cases may be appropriate.

III THE FOUR DIFFERENT CASES

Case 1 - The material follows Hooke's Law:

The volumetric relation is:

$$\sigma_m = 3K\varepsilon_m \tag{2}$$

where

$$3K = \frac{E}{1-2\nu}$$

The distortional relation follows as

$$\sigma_i - \sigma_m = 2G(\varepsilon_i - \varepsilon_m) \tag{3}$$

where

$$G = E/2(1+\nu)$$

From these equations we find

$$\sigma_r + 2\sigma_\theta = 3K(\varepsilon_r + 2\varepsilon_\theta) \tag{4}$$

$$\sigma_r - \sigma_\theta = 2G(\varepsilon_r - \varepsilon_\theta) \tag{5}$$

The strain displacement relations for infinitesimal strains are:

$$\varepsilon_r = du/dr \tag{6}$$

$$\varepsilon_\theta = u/r \tag{7}$$

From eq. (4) we obtain

$$\sigma_\theta = \tfrac{3}{2}K(\varepsilon_r + 2\varepsilon_\theta) - \tfrac{1}{2}\sigma_r \tag{8}$$

Substituting eq. (8) into eq. (5) we find

$$\sigma_r = \tfrac{4}{3}G(\varepsilon_r - \varepsilon_\theta) + K(\varepsilon_r + 2\varepsilon_\theta) \tag{9}$$

The equilibrium equation is:

$$\frac{d\sigma_r}{dr} + 2\,\frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{10}$$

By substitution of eqs. (5) and (9) into (10)

$$\frac{4}{3}G\left(\frac{d\varepsilon_r}{dr} - \frac{d\varepsilon_\theta}{dr}\right) + K\left(\frac{d\varepsilon_r}{dr} + 2\frac{d\varepsilon_\theta}{dr}\right) + 4\frac{G}{r}(\varepsilon_r - \varepsilon_\theta) = 0 \tag{11}$$

Then by substitution of eqs. (6) and (7) into (11)

$$\frac{d^2u}{dr^2} + \frac{2}{r}\,\frac{du}{dr} - 2\,\frac{u}{r^2} = 0 \tag{12}$$

The solution of eq. (12) is

$$u = Ar + Br^{-2} \tag{13}$$

where A and B are integration constants. Then from eqs. (6) and (7) we obtain

$$\varepsilon_r = A - \frac{2B}{r^3} \tag{14}$$

$$\varepsilon_\theta = A + \frac{B}{r^3} \tag{15}$$

and from eqs. (8) and (9) we have

$$\sigma_r = 3KA - 4GBr^{-3} \tag{16}$$

$$\sigma_\theta = 3KA + 2GBr^{-3} \tag{17}$$

Equations (13) to (17) give the displacement, strains and stresses in terms of the radius r and of the two integration constants A and B.

Case 2: The material is volumetrically elastic and distortionally locked: $\varepsilon_m < k_m$ and $\gamma_0 = k$.

The volumetric relation is

$$\sigma_m = 3K\varepsilon_m \tag{18}$$

The distortional relations are

$$\gamma_0 = k \tag{19}$$

and

$$\sigma_i - \sigma_m = \lambda \frac{\partial \gamma_0}{\partial (\varepsilon_i - \varepsilon_m)} \tag{20}$$

Now

$$\gamma_0 = \frac{2}{3}\left[(\varepsilon_r - \varepsilon_\theta)^2 + (\varepsilon_\theta - \varepsilon_\phi)^2 + (\varepsilon_\phi - \varepsilon_r)^2\right]^{1/2} \tag{21}$$

Hence eqs. (19) and (20) give

$$\gamma_0 = \frac{2\sqrt{2}}{3}(\varepsilon_\theta - \varepsilon_r) = k \tag{22}$$

from which

$$\varepsilon_\theta - \varepsilon_r = \frac{3k}{2\sqrt{2}} = k_1 \tag{23}$$

where k_1 is a new material constant.

Using eqs. (6) and (7), eq. (23) becomes

$$\frac{du}{dr} - \frac{u}{r} + k_1 = 0 \tag{24}$$

The solution of this first order differential equation is

$$u = C_1 r - k_1 r \ln r \tag{25}$$

Equation (25) gives the displacement. The strain components are

$$\varepsilon_r = C_1 - k_1 (1 + \ln r) \tag{26}$$

$$\varepsilon_\theta = C_1 - k_1 \ln r \tag{27}$$

The octahedral shearing strain can be written also as

$$\gamma_o = \frac{2}{\sqrt{3}} \left[(\varepsilon_r - \varepsilon_m)^2 + (\varepsilon_\theta - \varepsilon_m)^2 + (\varepsilon_\varphi - \varepsilon_m)^2 \right]^{1/2} \tag{28}$$

then

$$\frac{\partial \gamma_o}{\partial (\varepsilon_i - \varepsilon_m)} =$$

$$= \frac{2}{\sqrt{3}} \frac{\varepsilon_i - \varepsilon_m}{\left[(\varepsilon_r - \varepsilon_m)^2 + (\varepsilon_\theta - \varepsilon_m)^2 + (\varepsilon_\varphi - \varepsilon_m)^2 \right]^{1/2}} \tag{29}$$

or

$$\sigma_i - \sigma_m = \frac{\sqrt{2}\lambda}{k_1} (\varepsilon_i - \varepsilon_m) \tag{30}$$

because

$$\left[(\varepsilon_r - \varepsilon_m)^2 + (\varepsilon_\theta - \varepsilon_m)^2 + (\varepsilon_\varphi - \varepsilon_m)^2 \right]^{1/2} = \frac{\sqrt{3}}{2} k = \sqrt{\frac{2}{3}} k_1 .$$

Then from eqs. (30) we obtain

$$\sigma_r - \sigma_m = \frac{\sqrt{2}\lambda}{k_1}(\varepsilon_r - \varepsilon_m) \tag{31}$$

$$\sigma_\theta - \sigma_m = \frac{\sqrt{2}\lambda}{k_1}(\varepsilon_\theta - \varepsilon_m) \tag{32}$$

From eqs. (31) and (32) we find

$$\sigma_r - \sigma_\theta = \frac{\sqrt{2}\lambda}{k_1}(\varepsilon_r - \varepsilon_\theta) \tag{33}$$

The volumetric relation $\sigma_m = 3K\varepsilon_m$ can be written as

$$\sigma_r + 2\sigma_\theta = 3K(\varepsilon_r + 2\varepsilon_\theta) \tag{34}$$

From the last expression σ_θ is found to be

$$\sigma_\theta = \frac{3}{2}K(\varepsilon_r + 2\varepsilon_\theta) - \frac{1}{2}\sigma_r \tag{35}$$

Substituting eq. (35) into eq. (33) we obtain

$$\sigma_r = \frac{2\sqrt{2}\lambda}{3k_1}(\varepsilon_r - \varepsilon_\theta) + K(\varepsilon_r + 2\varepsilon_\theta) \tag{36}$$

Introducing eqs. (36) and (33) into the equilibrium equation

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{37}$$

we obtain

$$\frac{2\sqrt{2}}{3k_1}\left[(\varepsilon_r - \varepsilon_\theta)\frac{d\lambda}{dr} + \lambda\frac{d}{dr}(\varepsilon_r - \varepsilon_\theta)\right] + \frac{2\sqrt{2}\lambda}{k_1 r}(\varepsilon_r - \varepsilon_\theta) + K\frac{d}{dr}(\varepsilon_r + 2\varepsilon_\theta) = 0 \tag{38}$$

Using the strain-displacement relations (6) and (7) eq. (38)

becomes

$$\left(\frac{du}{dr}-\frac{u}{r}\right)\frac{d\lambda}{dr}+\left(\lambda+\frac{3}{2\sqrt{2}}k_1K\right)\left(\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}-\frac{2u}{r^2}\right)=0 \quad (39)$$

Separating variables

$$\frac{d\lambda}{\lambda+\frac{3}{2\sqrt{2}}k_1K}=\frac{\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}-\frac{2u}{r^2}}{\frac{u}{r}-\frac{du}{dr}}dr \quad (40)$$

Substituting eq. (24) to the right side of eq. (40) we obtain

$$\frac{d\lambda}{\lambda+\frac{3}{2\sqrt{2}}k_1K}=-3\frac{dr}{r} \quad (41)$$

and from equation (41) we find

$$\ln\left(\lambda+\frac{3}{2\sqrt{2}}k_1K\right)=-3\ln r+\ln C_2 \quad (42)$$

where $\ln C_2$ is the integration constant.

Then

$$\lambda=\frac{C_2}{r^3}-\frac{3}{2\sqrt{2}}k_1K \quad (43)$$

Thus from eq. (36) we obtain

$$\sigma_r = -\frac{2\sqrt{2}}{3}\frac{C_2}{r^3} + 3C_1K - 3Kk_1 \ln r \tag{44}$$

and from eq. (35) we obtain

$$\sigma_\theta = 3KC_1 - 3Kk_1 \ln r - \frac{3Kk_1}{2} + \frac{\sqrt{2}}{3}\frac{C_2}{r^3} \tag{45}$$

Case 3: The material is distortionally elastic and volumetrically locked: $\varepsilon_m = k_m$, $\gamma_0 < k$.

The volumetric relation is

$$\varepsilon_m = k_m$$

The distortional equation is

$$\sigma_i - \sigma_m = 2G(\varepsilon_i - \varepsilon_m)$$

As the material is volumetrically locked we obtain

$$\varepsilon_r + 2\varepsilon_\theta = 3k_m \tag{46}$$

which with the help of eqs. (6) and (7) gives the differential equation

$$\frac{du}{dr} + 2\frac{u}{r} - 3k_m = 0 \tag{47}$$

The solution of this equation is

$$u = Cr^{-2} + k_m r \tag{48}$$

Then from eqs. (6) and (7) we obtain

$$\varepsilon_r = k_m - 2Cr^{-3} \tag{49}$$

$$\varepsilon_\theta = k_m + Cr^{-3} \tag{50}$$

and as we have

$$\sigma_r - \sigma_\theta = 2G(\epsilon_r - \epsilon_\theta) \tag{51}$$

we obtain

$$\sigma_r - \sigma_\theta = -6GC r^{-3} \tag{52}$$

With the help of the equation of equilibrium we find

$$\frac{d\sigma_r}{dr} - \frac{12}{r} GC r^{-3} = 0 \tag{53}$$

Integrating we have

$$\sigma_r = C_0 - 4GC r^{-3} \tag{54}$$

then from eq. (52) we find

$$\sigma_\theta = C_0 + 2GC r^{-3} \tag{55}$$

Thus, we have the displacement u , strain components ϵ_r, ϵ_θ and stress components σ_r, σ_θ .

Case 4: The material is distortionally and volumetrically locked:

$$\epsilon_m = k_m \tag{56}$$

$$\gamma_0 = k \tag{57}$$

It can be shown that this case can not occur. Indeed, let us compare the expressions for the displacements which follow from the two equations:

$$\epsilon_m = k_m \tag{58}$$

$$\gamma_0 = k \tag{59}$$

They are

$$u = k_m r + C r^{-2} \tag{60}$$

$$u = k_1 r - C_1 r \ln r \tag{61}$$

The right hand terms of the eqs. (60) and (61) are not the same. Therefore this case can not occur.

IV REGIONS OF LOCKING

For a very small load we have case 1 at every point of the sphere. In this case

$$\varepsilon_\theta - \varepsilon_r = 3B r^{-3}$$

$$\varepsilon_m = \frac{\varepsilon_r + 2\varepsilon_\theta}{3} = A$$

It is seen that $\varepsilon_\theta - \varepsilon_r$ = max at r = a and that ε_m has the same value at all points of the hollow sphere.

Suppose that in the process of loading ε_m becomes equal to k_m before $\varepsilon_\theta - \varepsilon_r$ becomes equal to k. Then $\varepsilon_m = k_m$ at all points of the sphere simultaneously. Hence, volumetric locking will occur at all points at the same time, and consequently distortional locking cannot occur at any place.

If on the other hand, during the loading process $\varepsilon_\theta - \varepsilon_r$ becomes equal to k_1 at r = a, before ε_m becomes equal to k, then we shall have distortional locking at r= a, while the rest of the sphere is still elastic. By increasing the load, the region of distortional locking will expand at the expense of the elastic region, until either the entire sphere becomes distortionally locked or until for some value of the load the still remaining elastic region becomes volumetrically locked.

We shall consider now these possibilities analytically:

a) The entire sphere is elastic, then we have the case 1 over the entire sphere. Therefore,

$$u = Ar + Br^{-2} \tag{64}$$

$$\varepsilon_r = A - 2Br^{-3} \tag{65}$$

$$\varepsilon_\theta = A + Br^{-3} \tag{66}$$

$$\sigma_r = 3KA - 4GBr^{-3} \tag{67}$$

$$\sigma_\theta = 3KA + 2GBr^{-3} \tag{68}$$

b) The entire sphere is locked volumetrically, then we have over the entire sphere the case 3. Hence

$$u = k_m r + Cr^{-2} \tag{69}$$

$$\varepsilon_r = k_m - 2Cr^{-3} \tag{70}$$

$$\varepsilon_\theta = k_m + Cr^{-3} \tag{71}$$

$$\sigma_r = C_o - 4GCr^{-3} \tag{72}$$

$$\sigma_\theta = C_o + 2GCr^{-3} \tag{73}$$

c) The sphere is distortionally locked for r = a to r = t and it is elastic for r = t to r = b. Then we have case 2 for $a \leqq r \leqq t$ and case 1 for $t \leqq r \leqq b$.

Hence, for $a \leqq r \leqq t$

$$(u)_d = C_1 r - k_1 r \ln r \tag{74}$$

$$(\varepsilon_r)_d = C_1 - k_1(1 + \ln r) \tag{75}$$

$$(\varepsilon_\theta)_d = C_1 - k_1 \ln r \tag{76}$$

$$(\sigma_r)_d = -\frac{2\sqrt{2}}{3} C_2 r^{-3} + 3 C_1 K - 3K k_1 \ln r \tag{77}$$

$$(\sigma_\theta)_d = 3KC_1 - 3K k_1 \ln r - \frac{3K k_1}{2} + \frac{\sqrt{2}}{3} C_2 r^{-3} \tag{78}$$

and for $t \leqq r \leqq b$

$$(u)_e = Ar + Br^{-2} \tag{79}$$

$$(\varepsilon_r)_e = A - 2Br^{-3} \tag{80}$$

$$(\varepsilon_\theta)_e = A + Br^{-3} \tag{81}$$

$$(\sigma_r)_e = 3KA - 4GBr^{-3} \tag{82}$$

$$(\sigma_\theta)_e = 3KA + 2GBr^{-3} \tag{83}$$

with the continuity conditions

$$(\sigma_r)_d = (\sigma_r)_e \quad \text{at} \quad r = t \tag{84}$$

$$(u)_d = (u)_e \quad \text{at} \quad r = t \tag{85}$$

$$(\varepsilon_\theta)_e - (\varepsilon_r)_e = k_1 \quad \text{at} \quad r = t \tag{86}$$

d) The sphere is distortionally locked from r = a to r = t and volumetrically locked from r = t to r = b. Then we have case 2 for $a \leqq r \leqq t$ and case 3 for $t \leqq r \leqq b$.

For $a \leqq r \leqq t$

$$(u)_d = C_1 r - k_1 r \ln r \tag{87}$$

$$(\varepsilon_r)_d = C_1 - k_1 (1 + \ln r) \tag{88}$$

$$(\varepsilon_\theta)_d = C_1 - k_1 \ln r \tag{89}$$

$$(\sigma_r)_d = -\frac{2\sqrt{2}}{3} C_2 r^{-3} + 3C_1 K - 3Kk_1 \ln r \tag{90}$$

$$(\sigma_\theta)_d = 3KC_1 - 3Kk_1 \ln r - \frac{3Kk_1}{2} + \frac{\sqrt{2}}{3} C_2 r^{-3} \tag{91}$$

For $t \leq r \leq b$

$$(u)_v = k_m r + C r^{-2} \tag{92}$$

$$(\varepsilon_r)_v = k_m - 2C r^{-3} \tag{93}$$

$$(\varepsilon_\theta)_v = k_m + C r^{-3} \tag{94}$$

$$(\sigma_r)_v = C_0 - 4GC r^{-3} \tag{95}$$

$$(\sigma_\theta)_v = C_0 + 2GC r^{-3} \tag{96}$$

The continuity conditions are

$$(u)_d = (u)_v \quad \text{at} \quad r = t \tag{97}$$

$$(\sigma_r)_d = (\sigma_r)_v \quad \text{at} \quad r = t \tag{98}$$

$$(\varepsilon_\theta)_v - (\varepsilon_r)_v = k_1 \quad \text{at} \quad r = t \tag{99}$$

$$2(\varepsilon_\theta)_d + (\varepsilon_r)_d = 3k_m \quad \text{at} \quad r = t \tag{100}$$

V THE BOUNDARY CONDITIONS

The displacement of the inner and outer surfaces will be denoted by u_a and u_b, respectively. The four quantities u_a, u_b, σ_{ra}, σ_{rb} cannot be selected independently.

Indeed in cases (a) and (b) there are two constants to be determined; C_0 and C in case (b), and A and B in case (a). Hence only

two boundary conditions are required and are possible. Thus, the following six sets of boundary conditions are in order.

$u = u_a$	at	$r = a$	
$u = u_b$	at	$r = b$	I
$u = u_a$	at	$r = a$	
$\sigma_r = \sigma_{ra}$	at	$r = a$	II
$u = u_a$	at	$r = a$	
$\sigma_r = \sigma_{rb}$	at	$r = b$	III
$u = u_b$	at	$r = b$	
$\sigma_r = \sigma_{ra}$	at	$r = a$	IV
$u = u_b$	at	$r = b$	
$\sigma_r = \sigma_{rb}$	at	$r = b$	V
$\sigma_r = \sigma_{ra}$	at	$r = a$	
$\sigma_r = \sigma_{rb}$	at	$r = b$	VI

Similarly in case (c) we have five constants to determine - i.e. C_1, C_2, A, B, t - and three continuity conditions at $r = t$ are at our disposal. Hence, only two boundary conditions are required and possible. It will be seen in the next section that case (d) is a limiting case of (c). It follows that for case (d) the six sets of boundary conditions are in order.

VI THE SOLUTION OF THE PROBLEM

Case (a) Elastic Case:

In this case the constants A and B can be determined for any one of the above six sets of boundary conditions. As an example we consider

the set

$$u = u_a \quad \text{at} \quad r = a$$

$$\sigma_r = \sigma_{rb} \quad \text{at} \quad r = b$$

A and B are determined by using eqs. (64) and (67):

$$u_a = Aa + Ba^{-2} \tag{101}$$

$$\sigma_{rb} = 3KA - 4GBb^{-3} \tag{102}$$

from which

$$A = \frac{4G\frac{u_a}{a}\frac{a^3}{b^3} + \sigma_{rb}}{4G\frac{a^3}{b^3} + 3K} \tag{103}$$

$$B = \frac{\frac{u_a}{a}3K - \sigma_{rb}}{4G\frac{a^3}{b^3} + 3K}\, a^3 \tag{104}$$

Substituting A and B into eqs. (13), (16) and (17) we obtain the expressions

$$u = \frac{4G\frac{u_a}{a}\frac{a^3}{b^3} + \sigma_{rb}}{4G\frac{a^3}{b^3} + 3K}\, r + \frac{\frac{u_a}{a}3K - \sigma_{rb}}{4G\frac{a^3}{b^3} + 3K}\, a^3 r^{-2} \tag{105}$$

$$\sigma_r = \frac{4G\frac{u_a}{a}\frac{a^3}{b^3} + \sigma_{rb}}{\frac{4G}{3K}\frac{a^3}{b^3} + 1} - \frac{\frac{u_a}{a}3K - \sigma_{rb}}{\frac{a^3}{b^3} + \frac{3K}{4G}} a^3 r^{-3} \quad (106)$$

$$\sigma_\theta = \frac{4G\frac{u_a}{a}\frac{a^3}{b^3} + \sigma_{rb}}{\frac{4G}{3K}\frac{a^3}{b^3} + 1} + \frac{\frac{u_a}{a}3K - \sigma_{rb}}{2\frac{a^3}{b^3} + \frac{3K}{2G}} a^3 r^{-3} \quad (107)$$

These expressions can be written in dimensionless form in terms of the variable $(\frac{a}{r})$ as follows:

$$\frac{u}{u_a} = \frac{4G\frac{u_a}{a}\frac{a^3}{b^3} + \sigma_{rb}}{4G\frac{a^3}{b^3} + 3K}\frac{r}{a}\frac{a}{u_a} + \frac{\frac{u_a}{a}3K - \sigma_{rb}}{4G\frac{a^3}{b^3} + 3K}\frac{a}{u_a}\frac{a^2}{r^2} \quad (108)$$

$$\frac{\sigma_r}{G} = \frac{4\frac{u_a}{a}\frac{a^3}{b^3} + \frac{\sigma_{rb}}{G}}{\frac{4G}{3K}\frac{a^3}{b^3} + 1} - \frac{3\frac{K}{G}\frac{u_a}{a} - \frac{\sigma_{rb}}{G}}{\frac{3K}{4G} + \frac{a^3}{b^3}}\left(\frac{a}{r}\right)^3 \quad (109)$$

$$\frac{\sigma_\theta}{G} = \frac{4\frac{u_a}{a}\frac{a^3}{b^3} + \frac{\sigma_{rb}}{G}}{\frac{4G}{3K}\frac{a^3}{b^3} + 1} + \frac{3\frac{K}{G}\frac{u_a}{a} - \frac{\sigma_{rb}}{G}}{\frac{3K}{2G} + 2\frac{a^3}{b^3}}\left(\frac{a}{r}\right)^3 \quad (110)$$

Case (b): The entire sphere is volumetrically locked.

In this case the constants C_o and C can be determined for one of the six sets of boundary conditions except set (1). Indeed the expression of u, equation (69) includes only one constant and therefore either u_a or u_b would be sufficient to define this constant.

The volumetric locking will occur when the following condition is satisfied:

$$\varepsilon_r + 2\varepsilon_\theta = 3k_m$$

As an example let us consider again set (IV) in which $u = u_a$ at $r = a$, and $\sigma_r = \sigma_{rb}$ at $r = b$. In this case

$$u_a = k_m a + Ca^{-2} \quad (111)$$

from which substituting (111) into (69) we obtain

$$u = k_m\left(1 - \frac{a^3}{r^3}\right)r + \frac{a^2}{r^2}u_a \tag{113}$$

Using eq. (72) we obtain

$$\sigma_{rb} = C_o - 4GC\,b^{-3} \tag{114}$$

and with the help of eq. (111)

$$C_o = \sigma_{rb} + 4G\left(\frac{u_a}{a} - k_m\right)\left(\frac{a}{b}\right)^3 \tag{115}$$

is found. Substituting C and C_o into eqs. (72) and (73) it follows

$$\sigma_r = \sigma_{rb} + 4G\left(\frac{u_a}{a} - k_m\right)\left(\frac{a^3}{b^3} - \frac{a^3}{r^3}\right) \tag{116}$$

$$\sigma_\theta = \sigma_{rb} + 2G\left(\frac{u_a}{a} - k_m\right)\left(2\frac{a^3}{b^3} + \frac{a^3}{r^3}\right) \tag{117}$$

Writing eqs. (113), (116) and (117) in dimensionless form we find

$$\frac{u}{u_a} = k_m\left(1 - \frac{a^3}{r^3}\right)\left(\frac{r}{a}\right)\left(\frac{a}{u_a}\right) + \frac{a^2}{r^2} \tag{118}$$

$$\frac{\sigma_r}{G} = \frac{\sigma_{rb}}{G} + 4\left(\frac{u_a}{a} - k_m\right)\left(\frac{a^3}{b^3} - \frac{a^3}{r^3}\right) \tag{119}$$

$$\frac{\sigma_\theta}{G} = \frac{\sigma_{rb}}{G} + 2\left(\frac{u_a}{a} - k_m\right)\left(2\frac{a^3}{b^3} + \frac{a^3}{r^3}\right) \tag{120}$$

Volumetric locking will occur when in the previous case $A = k_m$. That is when

$$k_m = \frac{4G\,\frac{u_a}{a}\,\frac{a^3}{b^3} + \sigma_{rb}}{4G\,\frac{a^3}{b^3} + 3K} \tag{121}$$

This gives a relation between u_a and σ_{rb}. For a given u_a there corresponds a value of σ_{rb} for which volumetric locking over the entire sphere will occur.

Case (c): The sphere is distortionally locked at $a \leqq r \leqq t$ and elastic $t \leqq r \leqq b$.

We shall restrict our considerations to the case where at r=a, we have $u=u_a$ and at r=b, we have $\sigma_r=\sigma_{rb}$. Eq. (74) gives

$$u_a = C_1 a - k_1 a \ln a \tag{122}$$

From eq.(82) we obtain

$$\sigma_{rb} = 3KA - 4GBb^{-3} \tag{123}$$

Using the continuity conditions at r=t, that is eqs. (84), (85) and (86) we obtain

$$C_1 t - k_1 t \ln t = At + Bt^{-2} \tag{124}$$

$$-\frac{2\sqrt{2}}{3} C_2 t^{-3} + 3C_1 K - 3K k_1 \ln t = 3KA - 4GBt^{-3} \tag{125}$$

$$3Bt^{-3} = k_1 \tag{126}$$

The four unknowns C_1, C_2, A and B are determined from eqs. (122), (123), (124), (125) and (126) in terms of u_a, σ_{rb} and t.

$$A = \frac{u_a}{a} + k_1 \ln \frac{a}{t} - \frac{k_1}{3} \tag{127}$$

$$B = \frac{k_1 t^3}{3} \tag{128}$$

$$C_1 = \frac{u_a}{a} + k_1 \ln a \tag{129}$$

$$C_2 = \frac{3}{2\sqrt{2}} \left(K + \frac{4}{3}G\right) k_1 t^3 \tag{130}$$

The value of t will be determined from the following equation

$$\sigma_{rb} = 3K\left(\frac{u_a}{a} + k_1 \ln\frac{a}{t} - \frac{k_1}{3}\right) - \frac{4}{3}Gk_1\frac{t^3}{b^3} \tag{131}$$

Substituting eqs. (127), (128), (129) and (130) into eqs. (74), (77), (78), (79), (82) and (83) the following expressions are obtained.

For $a \leqq r \leqq t$

$$\frac{u}{u_a} = \frac{r}{a}\left(1 + k_1\frac{a}{u_a}\ln\frac{a}{r}\right) \tag{132}$$

$$\frac{\sigma_r}{G} = -\left(\frac{K}{G} + \frac{4}{3}\right)k_1\frac{t^3}{r^3} + 3\frac{K}{G}\left(\frac{u_a}{a} + k_1\ln\frac{a}{r}\right) \tag{133}$$

$$\frac{\sigma_\theta}{G} = 3\frac{K}{G}\left(\frac{u_a}{a} + k_1\ln\frac{a}{r} - \frac{k_1}{3}\right) + k_1\left(\frac{K}{G} + \frac{4}{3}\right)\frac{t^3}{2r^3} \tag{134}$$

For $t \leqq r \leqq b$

$$\frac{u}{u_a} = \left(1 + \frac{a}{u_a}k_1\ln\frac{a}{t}\right)\left(\frac{r}{a}\right) - \frac{1}{3}\frac{a}{u_a}k_1\frac{r}{a}\left(1 - \frac{t^3}{r^3}\right) \tag{135}$$

$$\frac{\sigma_r}{G} = 3\frac{K}{G}\left(\frac{u_a}{a} + k_1\ln\frac{a}{t} - \frac{k_1}{3}\right) - \frac{4}{3}k_1\frac{t^3}{r^3} \tag{136}$$

$$\frac{\sigma_\theta}{G} = 3\frac{K}{G}\left(\frac{u_a}{a} + k_1\ln\frac{a}{t} - \frac{k_1}{3}\right) + \frac{2}{3}k_1\frac{t^3}{r^3} \tag{137}$$

Distortional locking starts at r=a when t=a. Eq. (131) gives the relationship between u_a and σ_{rb} for which distortional locking will start:

$$\sigma_{rb} = 3K\left(\frac{u_a}{a} - \frac{k_1}{3}\right) - \frac{4G}{3}k_1\frac{a^3}{b^3} \tag{138}$$

Volumetric locking over the entire sphere can not occur when distortional locking has already occured over part of the sphere. Hence, whether volumetric locking will occur over the entire sphere first or distortional locking will occur at r=a first depends on which of the eqs. (121) or (138) will be valid first. Calculating ε_m for $t \leqq r \leqq b$ in case (c) one obtains the following relation:

$$\varepsilon_m = \frac{\varepsilon_\theta + 2\varepsilon_r}{3} = \frac{u_a}{a} + k_1 \ln\frac{a}{t} - \frac{k_1}{3} = A \tag{139}$$

Hence, ε_m is constant in the region $t \leqq r \leqq b$. In the region $a \leqq r \leqq t$ we obtain

$$\varepsilon_m = \frac{u_a}{a} + k_1 \ln\frac{a}{r} - \frac{k_1}{3} \tag{140}$$

Case (d): The sphere is distortionally locked for $a \leqq r \leqq t$ and volumetrically locked for $t \leqq r \leqq b$.

Again we restrict our considerations to the boundary conditions $r = a$, $u = u_a$, and $r = b$, $\sigma_r = \sigma_{r6}$. This case follows from case (c) by putting $\varepsilon_m = k_m$ for $t \leqq r \leqq b$. We obtain

$$\frac{u_a}{a} + k_1 \ln\frac{a}{t} - \frac{k_1}{3} = k_m \tag{141}$$

Expressions (132) to (137) continue to remain valid. Equation (141) is the condition under which case (d) is valid.

VII DISCUSSION

In Figure 6 we introduce the system of rectangular cartesian coordinates u/u_a, σ_{rt}/G . The plane $u/u_a, \sigma_{rt}/G$ can be divided in regions according to the type of behavior of the hollow sphere. We shall have four regions in the plane corresponding to the cases (a), (b), (c), and (d). Our problem will be to determine these regions. Although the discussion refers to any values of the quantities K/G, a/b,

$k_m < 0$, and $k_1 > 0$, the regions shown in Figure 6 are calculated for the specific numerical values $K/G = 2$, $a/b = 0.50$, $k_m = -0.2$, $k_1 = 1$.

For volumetric locking to occur over the entire sphere we must have

$$4\frac{a^3}{b^3}\frac{u_a}{a} + \frac{\sigma_{rb}}{G} - k_m\left(4\frac{a^3}{b^3} + 3\frac{K}{G}\right) = 0 \tag{142}$$

which follows from condition (121).

Condition (142) represents a straight line the slope of which is given by the ratio $-4a^3/b^3$. Therefore, it is seen that the slope of this straight line is always negative and depends on the ratio a/b only. The intersections of this straight line with the two axes have the co-ordinates

$$\frac{u_a}{a} = k_m\left(1 + \frac{3}{4}\frac{K}{G}\frac{b^3}{a^3}\right) \quad \text{for} \quad \frac{\sigma_{rb}}{G} = 0 \tag{143}$$

$$\frac{\sigma_{rb}}{G} = k_m\left(4\frac{a^3}{b^3} + 3\frac{K}{G}\right) \quad \text{for} \quad \frac{u_a}{a} = 0$$

It is seen that this line intersects the two axes at negative values of the variables.

For the specific numerical values considered line (142) has the equation

$$\frac{u_a}{a} + 2\frac{\sigma_{rb}}{G} + 2.60 = 0 \tag{144}$$

as is shown in Fig. 6 as line FC.

For distortional locking to occur over the region $a \leq r \leq t$ of the hollow sphere it is necessary that the condition

$$3\frac{K}{G}\frac{u_a}{a} - \frac{\sigma_{rb}}{G} + 3\frac{K}{G}k_1 \ln\frac{a}{t} - \frac{K}{G}k_1 - \frac{4}{3}k_1\frac{t^3}{b^3} = 0 \tag{145}$$

will be satisfied. This condition follows from equation (131). It represents a straight line of slope 3 K/G. The position of this line depends on the value of t. For t = a we have

$$3\frac{K}{G}\frac{u_a}{a} - \frac{\sigma_{rb}}{G} - \frac{K}{G}k_1 - \frac{4}{3}k_1\frac{a^3}{b^3} = 0 \tag{146}$$

while for t = b we obtain

$$3\frac{K}{G}\frac{u_a}{a} - \frac{\sigma_{rb}}{G} + 3\frac{K}{G}k_1 \ln\frac{a}{b} - \frac{K}{G}k_1 - \frac{4}{3}k_1 = 0 \tag{147}$$

These are the two limiting positions of line (145). The one position corresponds to incipiant distortional locking, while the other position corresponds to complete distortional locking. By changing the value of t line (145) moves parallel to itself.

For the specific numerical values considered line (145) has the equation

$$6\frac{u_a}{a} - \frac{\sigma_{rb}}{G} + 6\ln\frac{a}{t} - 2 - \frac{4}{3}\frac{t^3}{b^3} \tag{148}$$

The two limiting positions of line (145) have the equations

$$6\frac{u_a}{a} - \frac{\sigma_{rb}}{G} - 2.166 = 0 \tag{149}$$

and

$$6\frac{u_a}{a} - \frac{\sigma_{rb}}{G} - 7.4886 = 0 \tag{150}$$

and are shown in Fig. 6 as lines BC and ED.

For a given t the elastic region will lock volumetrically when

$$\frac{u_a}{a} = k_m + \frac{k_1}{3} - k_1\ln\frac{a}{t} \tag{151}$$

Hence, equations (145) and (148) give

$$\frac{\sigma_{r\theta}}{G} = 3\frac{k}{G}k_m - \frac{4}{3}k_1\frac{t^3}{b^3} \tag{152}$$

Equations (151) and (152) are the parametric equations of the line representing distortional locking for $r \leq t$ and volumetric locking for $r \geq t$. For $t = a$ this curve passes by the intersection of the lines (146) and (142). The slope of this line is given by $-4t^3/b^3$. It is seen that this line is tangent to the line (142) at $t = a$ and has a slope equal to -4 at $t = b$.

For the specific numerical values considered line (151), (152) has the equations

$$\begin{aligned}\frac{u_a}{a} &= 0.133 - \ln\frac{a}{t} \\ \frac{\sigma_{r\theta}}{G} &= -1.2 - \frac{4}{3}\frac{t^3}{b^3}\end{aligned} \tag{153}$$

and is shown in Fig. 6 as line CD.

In this discussion we assume $k_1 > 0$ and $k_m < 0$. The second assumption does not need any justification in view of the fact that we consider volumetric locking to occur only when the decrease in volume per unit volume reaches a certain limiting value. On the other hand it is necessary to prove that k_1 is indeed positive for the type of boundary conditions selected in this problem. This can be shown by considering the elastic range. Equations (65) and (66) give

$$\varepsilon_\theta - \varepsilon_r = \frac{3B}{r^3} \tag{154}$$

which together with equations (104) and (23) show that when distortional locking occurs

$$k_1 = 3\left(\frac{u_a}{a}3K - \sigma_{r\theta}\right)\frac{a^3}{r^3}\Big/\left(4G\frac{a^3}{b^3} + 3K\right) \tag{155}$$

Therefore, when

$$\frac{u_a}{a} > \frac{\sigma_{v\ell}}{3K} \tag{156}$$

we have $k_1 > 0$. Condition (156) is always valid when $\sigma_{v\ell} < 0$ and $u_a > 0$.

In Figure 6 the region OFCD represents the values of the load for which the hollow sphere is completely elastic (restricting ourselves to the case $u_a/a > 0, \sigma_{v\ell}/G < 0$). Line FC represents the values of the load for which the entire sphere is volumetrically locked. Line CB represents the condition of incipiant distortional locking. Line ED represents the condition of complete distortional locking of the sphere. Line CD represents the condition of complete locking -- partly distortional, partly volumetric. Region BCDE represents the values of the load for which the hollow sphere is partly distortionally locked and partly elastic. Finally line AG represents the condition for which the portion $a \leq r \leq \frac{a+b}{2}$ of the sphere is distortionally locked whereas the remainder of the sphere is elastic. Of course the point G represents the case when the above mentioned elastic portion of the sphere becomes volumetrically locked.

It is interesting to see that when the loading condition of the sphere is represented by a point of the line FCD an increase of u_a/a without an appropriate increase in $\sigma_{v\ell}/G$ will revert the sphere, at least partly, to the elastic condition. The ratio $\frac{d(u_a/a)}{d(\sigma_{v\ell}/G)}$ necessary for keeping the sphere volumetrically locked is constant along the line FC and it depends on the ratio a/b only. The above ratio for keeping the sphere locked - partly distortionally and partly volumetrically - is variable along the line CD; it decreases from the previous constant value to -4.

The question arises of what happens when the ratio $\frac{d(u_a/a)}{d(\sigma_{r\theta}/G)}$ is such that the loading condition of the sphere tends to move below the line FCD or to the right or below the line DE. In these two cases the boundary value problem changes because when the sphere is completely locked as is on the lines FCD and DE both values u_a and u_b are fixed and the problem becomes one of finding the stress distribution for given values of u_a and u_b. This problem will not be discussed in this report.

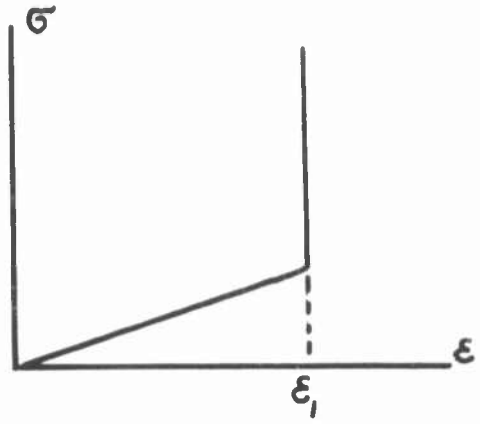

FIG. 1

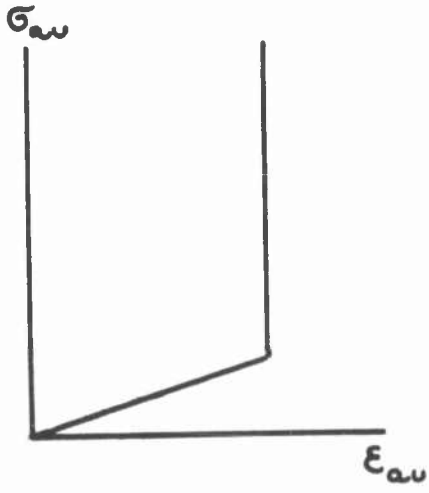

FIG. 2a

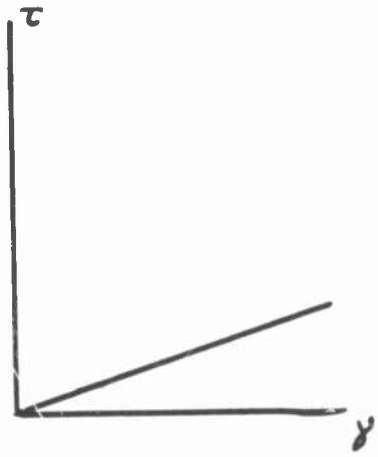

FIG. 2b

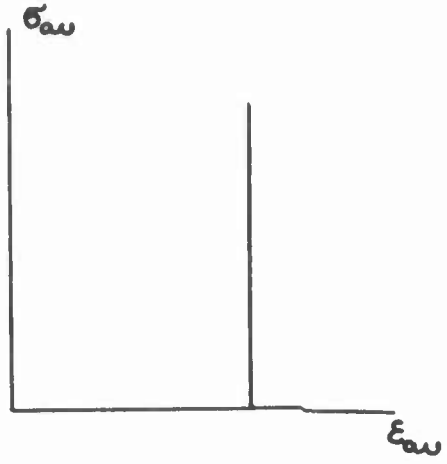

FIG. 3a

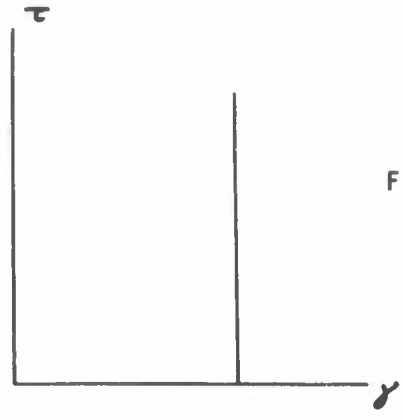

FIG. 3b

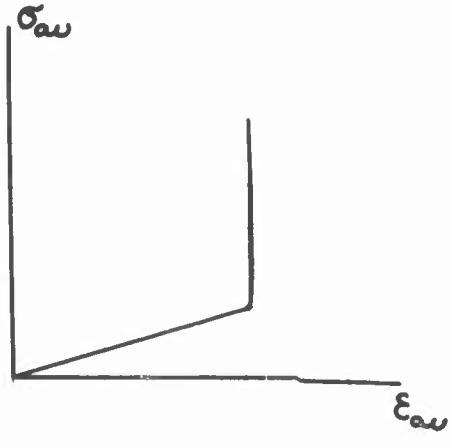

FIG. 4a

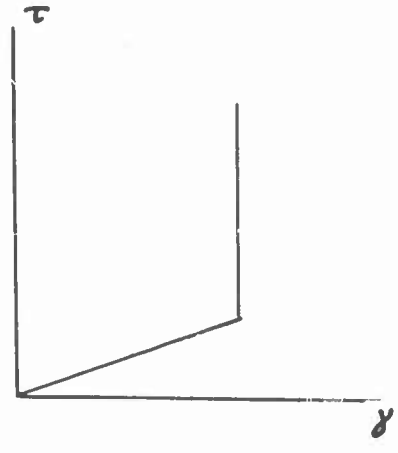

FIG. 4b

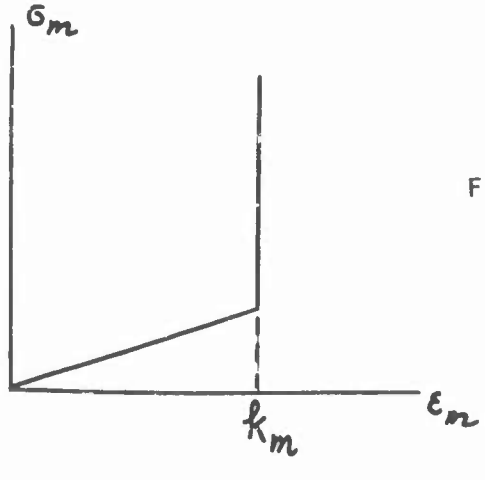

FIG. 5

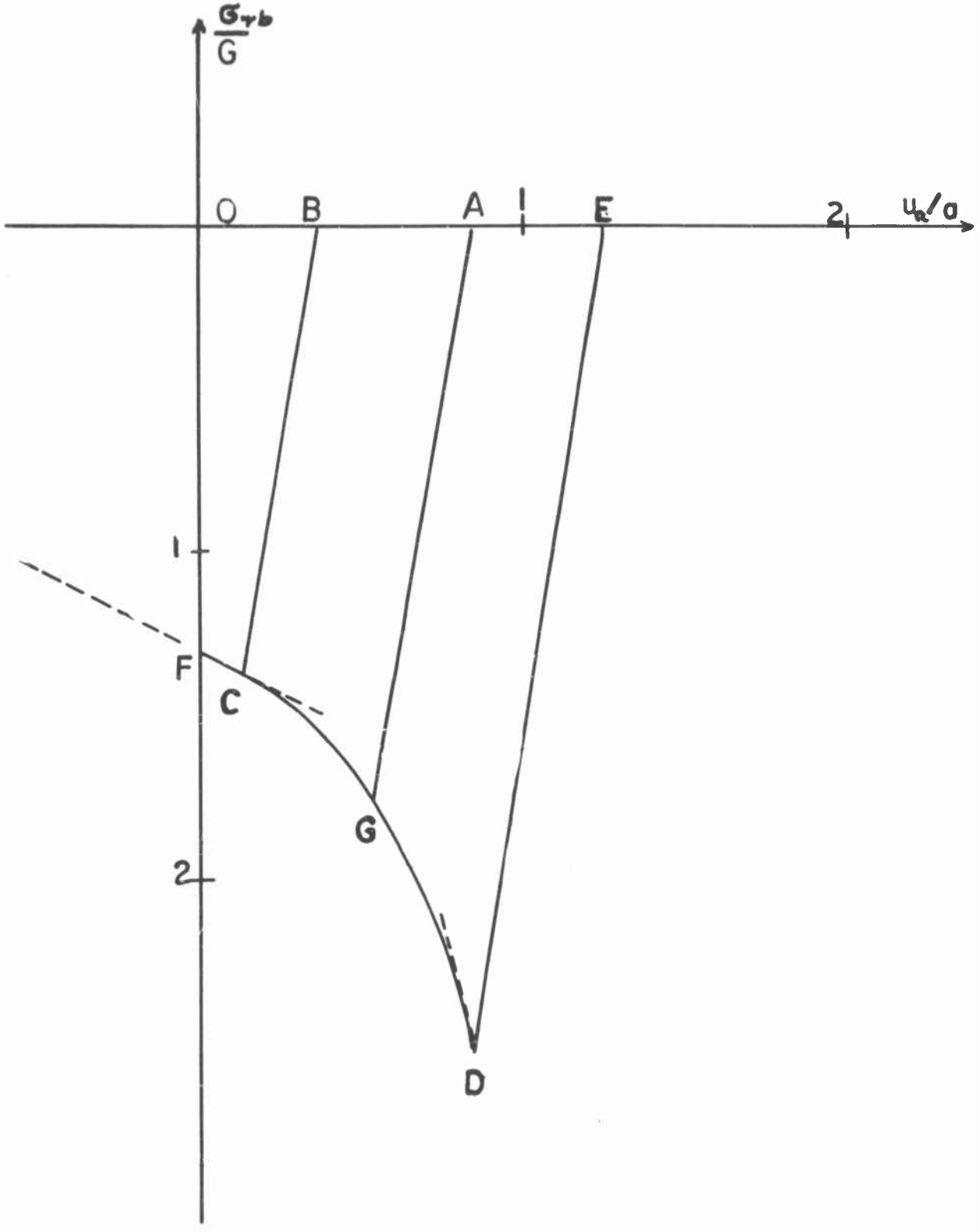

FIG. 6

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